Chapter 2: Probability, Statistics, and Traffic Theories

P2.1. A random number generator produces numbers between 1 and 99. If the current value of the random variable is 45, then what is the probability that the next randomly generated value for the same random variable, will also be 45. Explain clearly.

[Solution]
Let the random variable be \( X \) such that \( 1 \leq X \leq 99 \).
Current value of \( X \) is 45.
Since the value of generation of random numbers is independent of the previous generation, the probability of generating 45 again is the same as for the first time. Hence,

\[
P(X = 45) = \frac{1}{99}
\]

P2.2. A random digit generator on a computer is activated three times consecutively to simulate a random three-digit number.
(a) How many random three-digit numbers are possible?
(b) How many numbers will begin with the digit 2?
(c) How many numbers will end with the digit 9?
(d) How many numbers will begin with the digit 2 and end with the digit 9?
(e) What is the probability that a randomly formed number ends with 9 given that it begins with a 2?

[Solution]
(a) 900
(b) 100
(c) 90
(d) 10
(e) 1/10

P2.3. A snapshot of the traffic pattern in a cell with 10 users of a wireless system is given as follows:

<table>
<thead>
<tr>
<th>User Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Initiation Time</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Call Holding Time</td>
<td>5</td>
<td>7</td>
<td>4</td>
<td>8</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) Assuming the call setup/connection and call disconnection time to be zero, what is the average duration of a call?
(b) What is the minimum number of channels required to support this sequence of calls?
(c) Show the allocation of channels to different users for part (b) of the Problem.
(d) Given the number of channels obtained in part (b), for what fraction of time are the channels utilized?

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[Solution]

(a) Average duration of a call is 
\[ \frac{5748621432}{10} \]

(b) By plotting the number of calls by all users, we can determine how many users need to have a channel simultaneously. This gives us the minimum number of channels required to support the sequence of calls as 6.

(c) Allocation of channels to various users is

<table>
<thead>
<tr>
<th>Channel number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>User number</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>10</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>User number (allocated to same channel)</td>
<td>9</td>
<td>7</td>
<td>4</td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Total duration of the calls is 42.
   Total amount of time channels are available is 1166.
   Therefore fraction of time channels are used is \( \frac{42}{66} = 0.6363 \).

P2.4. A department survey found that 4 of 10 graduate students use CDMA cell phone service. If 3 graduate students are selected at random, what is the probability that 3 graduate students use CDMA cell phones?

[Solution]

\[ \frac{C_3^4}{C_{10}^3} \frac{2}{60} = 0.0333. \]

P2.5. There are three red balls and seven white balls in box A, and six red balls and four white balls in box B. After throwing a die, if the number on the die is 1 or 6, then pick a ball from box A. Otherwise, if any other number appears (i.e., 2, 3, 4, or 5), then pick a ball from box B. Selected ball has to put it back before proceeding further. Answer the followings:

(a) What is the probability that the selected ball is red?
(b) What is the probability a white ball is picked up in two successive selections?

[Solution]

(a) Probability of selecting the box \( \times \) probability of selecting a red ball is equal to 
\[ \frac{2}{6} \frac{3}{10} \frac{4}{6} = 0.5. \]

(b) Since we are replacing the balls, the occurrence of two successive white balls is independent.
   Thus probability of picking two white balls in succession is equal to 
\[ \frac{2}{7} \frac{4}{4} \frac{2}{7} \frac{4}{4} = 0.25. \]
P2.6. Consider an experiment consisting of tossing two true dice. Let $X$, $Y$, and $Z$ be the numbers shown on the first die, the second die, and total of both dice, respectively. Find $P (X \leq 1, Z \leq 2)$ and $P (X \leq 1) P (Z \leq 2)$ to show that $X$ and $Y$ are not independent.

[Solution]
\[
P(X, Y, Z) = \frac{1}{36}
\]
\[
P(X \leq 1, Z \leq 2) = \frac{1}{36}
\]
\[
P(X \leq 1) P(Z \leq 2) = \frac{1}{6} \cdot \frac{1}{36}
\]
Since $P(X \leq 1, Z \leq 1)$ and $P(X \leq 1) P(Z \leq 1)$ are not equal, they are not independent.

P2.7. The following table shows the density of the random variable $X$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>0.03</td>
<td>0.01</td>
<td>0.04</td>
<td>0.3</td>
<td>0.3</td>
<td>0.1</td>
<td>0.07</td>
<td>?</td>
</tr>
</tbody>
</table>

(a) Find $p(8)$
(b) Find the table for CDF $F(x)$
(c) Find $P(3 \leq X \leq 5)$
(d) Find $P(X \leq 4)$ and $P(X < 4)$. Are the probabilities the same? (e) Find $F(-3)$ and $F(10)$

[Solution]
(a) $p(8) = 0.15$

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>0.03</td>
<td>0.01</td>
<td>0.04</td>
<td>0.3</td>
<td>0.3</td>
<td>0.1</td>
<td>0.07</td>
<td>0.15</td>
</tr>
</tbody>
</table>

(b) $F(x)$ is given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>0.03</td>
<td>0.04</td>
<td>0.08</td>
<td>0.38</td>
<td>0.68</td>
<td>0.78</td>
<td>0.85</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(c) $P(3 \leq X \leq 5) = 0.04 + 0.3 + 0.3 = 0.68$ (d)
$P(X \leq 4) = 0.38$ and $P(X < 4) = 0.08$
Thus they are not equal.
(e) $F(-3) = 0$ and $F(10) = 1$

P2.8. The density for $X$ is given in the table of Problem 7. (a)
Find $E[X]$
(b) Find $E[X^2]$ (c)
Find $Var[X]$
(d) Find the standard deviation for $X$. 
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[Solution]

(a) 

\[
\begin{array}{cccccccc}
  E & X & 1 & 0.03 & 2 & 0.01 & 3 & 0.04 & 4 & 0.3 & 5 & 0.3 & 6 & 0.1 \\
  7 & 0.07 & 8 & 0.15 \\
\end{array}
\]

E X 2 = 8 \sum_{x=1}^{8} x^2 p x = 29.36.

(c) \text{Var}[X] = 2.7344.

(d) Standard deviation = 1.65.

P2.9. Find the probability when

(a) \( k = 2 \) and \( \lambda = 0.01 \) for Poisson distribution.

(b) \( p = 0.01 \) and \( k = 2 \) for geometric distribution.

(c) Repeat (b) when binomial distribution is used and \( n = 10 \).

[Solution]

(a) \( p_k = \frac{k}{\lambda!} e^{-\lambda} \), when \( k = 2, \lambda = 0.01 \), Probability = 0.0000495025.

(b) \( p_k = p(1-p)^k \), when \( k = 2, \lambda = 0.01 \), Probability = 0.0000495025.

(c) \( p_k = \frac{k}{n!} p^k (1-p)^{n-k} \) when \( n = 10, k = 2, = 0.01 \), Probability = 0.00415235.

P2.10. Find the distribution function of the maximum of a finite set of independent random variables \( \{X_1, X_2, \ldots, X_n\} \), where \( Xi \) has distribution function \( F_{Xi} \). What

is this distribution when \( X_i \) is exponential with a mean of \( \frac{1}{\mu} \).

[Solution]

\[
Y = \max(X_1, X_2, \ldots, X_n)
\]

\[
P (Y \leq y) = P \max(X_1, X_2, \ldots, y)
\]

\[
P(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y)
\]

\[
P_X_i (y)
\]

\[
\frac{1}{\mu}
\]

\[
F_{X_i} (y)
\]
When $X_i$ is exponential with mean $\lambda$, 
\[ F_{X_i}(y) = 1 - e^{-\lambda y}. \]

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Therefore, the CDF of $Y$ is given by
\[ F_Y(y) = \sum_{i=1}^{n} F_{X_i}(y) = \sum_{i=1}^{n} 1 - e^{-\lambda y}. \]

P2.11. The number of calls arrive under a particular time in a cell has been established to be a Poisson distribution. The average number of calls arriving in a cell in 1 millisecond is 5. What is the probability that 8 calls arrive in a cell in a given millisecond?

[Solution]
Since $\lambda = 5$,
\[ p_k = \frac{k e^{-\lambda}}{k!} = \frac{5^k e^5}{k!}, \]
so
\[ 8! 0.065278. \]

P2.12. Given that the number of arrivals of data packet in the receiver follows a Poisson distribution on which arrival rate is 10 arrivals per second. What is the probability that the number of arrivals is more than 8 but less than 11 during a time of interval of 2 seconds?

[Solution]
The number of arrivals during a time interval of 2 seconds follows a Poisson distribution with mean $10 \times 2 = 20$ arrivals/sec. Hence,
\[ P(9, 10) \]
\[ = \frac{20^9 e^{20}}{9!} - \frac{20^{10} e^{20}}{10!} \]
\[ = 0.00872446. \]

P2.13. In a wireless office environment, all calls are made between 8 am and 5 pm over the period of 24 hours. Assuming the number of calls to be uniformly distributed between 8 am and 5 pm, find the pdf of the number of calls over the 24 hour period. Also, determine the CDF and the variance of the call distribution.

[Solution]
Since
\[ f(x) = \frac{1}{9}, \quad x = 8, 17 \]
\[ f(x) = 0, \quad x = 0, 8 \quad \cup 17, 24 \]
the CDF is given by
\[ F_X(x) = \begin{cases} 0, & x < 0.8 \\ \frac{8}{1}, & 8.17 \leq x < 17 \\ 9, & x \geq 17.24 \end{cases} \]

Therefore,
\[ E X = \int_8^{17} x f(x) dx \]
\[ = \frac{17}{8} \frac{1}{9} \]
\[ = 25 \frac{1}{2} \]
\[ = 12.5, \]
\[ E X^2 = \int_8^{17} x^2 f(x) dx \]
\[ = \frac{17}{8} \frac{2}{9} \]
\[ = 163, \]
\[ Var X = E X^2 - (E X)^2 = 163 - (12.5)^2 \]
\[ = 6.75. \]

**P2.14.** A gambler has a regular coin and a two-headed coin in his packet. The probability of selecting the two-head coin is given as \( p = \frac{2}{3} \). He selects a coin and flips it \( n = 2 \) times and obtains heads both the times. What is the probability that the two-headed coin is picked both the times?

**[Solution]**
Probability is equal to
\[ \frac{2}{3} \cdot \frac{2}{1} \cdot \frac{4}{3} = \frac{16}{9} \]

**P2.15.** A Poisson process exhibits a memoryless property and is of great importance in traffic analysis. Prove that this property is exhibited by all Poisson processes. Explain clearly every step of your analytical proof.

**[Solution]**
In order to prove that Poisson process is memoryless, we need to prove the following equation:
\[ P X \mid X \quad P X \mid \]
Thus,

$$
\begin{align*}
PXt & \quad |X \quad P X t \quad \bigcap X \\
& \quad P X \\
& \quad 1 PXt \\
& \quad 1 PX \\
& \quad 1 PXt \\
& \quad PX t.
\end{align*}
$$

P2.16. What should be a relationship between call arrival rate and service rate when a cellular system is in a steady state? Explain clearly.

**[Solution]**

When the system reaches the steady state, the call arrival rate should be less or equal to the service rate, otherwise the system will be unstable.

P2.17. Consider a cellular system with an infinite number of channels. In such a system, all arriving calls begin receiving service immediately. The average call holding time is \( \frac{1}{n} \) when there are \( n \) calls in the system. Draw a state transition diagram for this system and develop expressions for the following:

(a) Steady-state probability \( P_n \) of \( n \) calls in the system.
(b) Steady-state probability \( P_0 \) of no calls in the system.
(c) Average number of calls in the system, \( L_s \).
(d) Average dwell time, \( W_s \).
(e) Average queue length, \( L_Q \).

**[Solution]**

The system model is \( M/M/\infty/\infty \) with average arrival rate of \( \lambda \) and average service rate of \( \mu/n \).

Probability \( P_i \) for the system with \( n \) calls can be found through the system balance equations as follows:

\[
P_i = P_{i-1} \lambda \mu \quad \text{for} \quad i > 0, \quad P_0 = \frac{1}{\lambda}.
\]

Thus, we have

\[
P_i = \frac{1}{\lambda} \mu, \quad \text{for} \quad i > 0.
\]

Using normalized condition

we have
\[ P_{i} \mid 1, i_{0} \]
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\[ P_0 = \frac{1}{i!} \]

(a) Steady-state probability \( P_n \) of \( n \) calls in the system is given by

\[ nP \frac{1^n}{n!} e^{-n}, \quad 0 \leq n. \]

(b) Steady-state probability \( P_0 \) of no calls in the system is given by

\[ P_0 = e^{-i}. \]

(c) Average number of calls in the system \( L_S \) is given by

\[ L_S = \frac{iP_i}{i}. \]

(d) Average dwell time \( W_S \) is given by

\[ W_S = \frac{L_S}{1}. \]

(e) Average queue length is equal to 0.

P2.18. Consider a cellular system in which each cell has only one channel (single server) and an infinite buffer for storage the calls. In this cellular system, call arrival rates are discouraged, that is, the call rate is only \( \lambda/\left(n + 1\right) \) when there are \( n \) calls in the system. The interarrival times of calls are exponentially distributed. The call holding times are exponentially distributed with mean rate \( \mu \). Develop expressions for the following:

(a) Steady-state probability \( P_n \) of \( n \) calls in the system.
(b) Steady-state probability \( P_0 \) of no calls in the system.
(c) Average number of calls in the system, \( L_S \).
(d) Average dwell time, \( W_S \).
(e) Average queue length, \( L_Q \).

[Solution]

The system model is M/M/1/∞ with average arrival rate of \( \lambda/ \left(n + 1\right) \), and average service rate of \( \mu \).

Traffic intensity \( \frac{n-1}{n+1} \).
Chapter 2: Probability, Statistics, and Traffic Theories

Probability \( P_n \) for the system with \( n \) calls can be found through the system balance equations as follows:

\[
P_i = \frac{1}{i!} P_i, \quad 0 \leq i.
\]

Thus, we have

\[
P_i = \frac{1}{i!} P_i, \quad 0 \leq i.
\]

Using normalized condition

\[
P_0 = 1,
\]
we have

\[
P_0 = \frac{1}{i!} e^i.
\]

(a) Steady-state probability \( P_n \) of \( n \) calls in the system is given by

\[
P_n = \frac{n^n}{n!} e^{-n}, \quad 0 \leq n.
\]

(b) Steady-state probability \( P_0 \) of no calls in the system is given by

\[
P_0 = e^0.
\]

(c) Average number of calls in the system \( L_S \) is given by

\[
L_S = \sum_{i=1}^{\infty} i P_i
\]

(d) Average arrival rate is

\[
\bar{\lambda} = \sum_{i=0}^{\infty} i P_i = e^{-\mu}.
\]

Therefore, average dwell time \( W_S \) is given by

\[
W = \frac{L_S}{\mu}.
\]
(e) Since average queue waiting time is given by
\[ W_q = \frac{1}{s} e^{-e} \]
and average queue length is given by
\[ L_q = \frac{1}{e} \]

P2.19. In a transition diagram of M/M/5 model, write the state transition equations and find a relation for the system to be in each state.

[Solution]
Hint: The system model is M/M/5 with average arrival rate, and average service rate of
\[ n, \quad n \ 1, 2, 3, 4, \]
\[ 5, \quad n \ 5, \]
The probability of \( n \) jobs in the system is given by
\[ P_n = \frac{1}{n!} P_0, \quad n \ 1, 2, 3, 4, \]
\[ \frac{1}{5!5^n} P_0, \quad n \ 5, \]
P2.20. In the M/M/1/ queueing system, suppose and are doubled. How are \( L_s \) and \( W_s \) changed?

**Solution** \( L_s \) is the same as M/M/1/ and \( W_s \) is the half of M/M/1/.

P2.21. If the number of mechanics in Example 2.4 is increased to 3, what is the impact on the performance parameters?

**Solution** It can be seen that the system is M/M/3/\( \infty \) queueing system with \( \lambda = 3/m \), and the service rate \( \mu = \frac{60}{10} = 6/m \). Therefore, the offered load \( \rho = \frac{3}{6} = \frac{1}{2} \).
Chapter 2: Probability, Statistics, and Traffic Theories
Chapter 2
Outline

Introduction

Probability Theory and Statistics Theory
- Random variables
- Probability mass function (pmf)
- Probability density function (pdf)
- Cumulative distribution function (CDF)

Expected value, nth moment, nth central moment, and variance
- Some important distributions

Traffic Theory
- Poisson arrival model, etc.

Basic Queuing Systems
- Little’s law
- Basic queuing models
Introduction

Several factors influence the performance of wireless systems:
- Density of mobile users
- Cell size
- Moving direction and speed of users (Mobility models)
- Call rate, call duration
- Interference, etc.

Probability, statistics theory and traffic patterns, help make these factors tractable
Probability Theory and Statistics Theory

Random Variables (RVs)

Let $S$ be sample associated with experiment $E$
$X$ is a function that associates a real number to each $s \in S$
RVs can be of two types: Discrete or Continuous
Discrete random variable $\Rightarrow$ probability mass function (pmf)
Continuous random variable $\Rightarrow$ probability density function (pdf)

Discrete random variable
Continuous random variable
Discrete Random Variables

In this case, $X(s)$ contains a finite or infinite number of values.

The possible values of $X$ can be enumerated.

**For Example:** Throw a 6 sided dice and calculate the probability of a particular number appearing.

- Probability
  - 2: 0.3
  - 3: 0.1
  - 5: 0.2
  - 6: 0.2

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Discrete Random Variables

The probability mass function (pmf) $p(k)$ of $X$ is defined as:

$$p(k) = p(X = k), \quad \text{for } k = 0, 1, 2, \ldots$$

where

1. Probability of each state occurring
   $$0 \leq p(k) \leq 1, \quad \text{for every } k;$$
2. Sum of all states
   $$p(k) = 1, \quad \text{for all } k$$
Continuous Random Variables

In this case, $X$ contains an infinite number of values

Mathematically, $X$ is a continuous random variable if there is a function $f$, called probability density function (pdf) of $X$ that satisfies the following criteria:

1. $f(x) \geq 0$, for all $x$;

2. $\int f(x)dx = 1$
Cumulative Distribution Function

Applies to all random variables
A cumulative distribution function (CDF) is defined as:
For discrete random variables:

\[ P(k) = P(X = k) = \sum_{x=k}^{\infty} f(x) \]

For continuous random variables:

\[ F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x) \]
The pdf $f(x)$ of a continuous random variable $X$ is the derivative of the CDF $F(x)$, i.e.,

$$f(x) = \frac{dF_X(x)}{dx}$$
Expected Value, $n^{th}$ Moment, $n^{th}$ Central Moment, and Variance

Average $E[X] = \sum_{k} kP(X = k) = 1/6$

$E[X] = 0.166$

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Expected Value, $n^{th}$ Moment, $n^{th}$ Central Moment, and Variance

Discrete Random Variables

Expected value represented by $E$ or average of random variable

$$E[X] = \sum_{k} k P(X = k)$$

Variance or the second central moment

$$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

$n^{th}$ moment

$$E[X^n] = \sum_{k} k^n P(X = k)$$

$n^{th}$ central moment

$$E[(X - E[X])^n] = \sum_{k} (k - E[X])^nP(X = k)$$
**Expected Value, n\text{th} Moment, n\text{th} Central Moment, and Variance**

**Continuous Random Variable**

Expected value or mean value
\[ E[X] = \int xf(x) \, dx \]

Variance or the second central moment
\[ ^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2 \]

\(n\text{th}\) moment
\[ E[X^n] = \int x^n f(x) \, dx \]

\(n\text{th}\) central moment
\[ E[(X - E[X])^n] = \int (x - E[X])^n f(x) \, dx \]
Some Important Discrete Random Distributions

**Poisson**

\[ P(X = k) = \frac{k^e}{k!}, \quad k = 0, 1, 2, \ldots, \text{and} \quad 0 \]

\[ E[X] = , \quad \text{and} \quad Var(X) = \]

**Geometric**

\[ P(X = k) = p(1-p)^{k-1} \]

where \( p \) is success probability

\[ E[X] = 1/(1-p), \quad \text{and} \quad Var(X) = p/(1-p)^2 \]
Some Important Discrete Random Distributions

Binomial

Out of $n$ dice, exactly $k$ dice have the same value: probability $p^k$ and $(n-k)$ dice have different values: probability $(1-p)^{n-k}$.

For any $k$ dice out of $n$:

\[ P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} , \]

where,

- $k = 0, 1, 2, \ldots, n$;
- $n = 0, 1, 2, \ldots$;
- $p$ is the success probability, and
- $n!$ is the factorial of $n$.

\[ k! (nk)! \]

http://en.wikipedia.org/wiki/
Some Important Continuous Random Distributions

Normal: $E[X] = \mu$, and $Var(X) = \sigma^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty$$

and the cumulative distribution function can be obtained by

$$F_X(x) = \frac{1}{2} \int_{-\infty}^{x} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

http://en.wikipedia.org/wiki/
Some Important Continuous Random Distributions

**Uniform**

\[
\begin{align*}
\frac{d}{b-a}, & \text{ for } a < x < b \\
f_X(x) & = \begin{cases} 
    b, & \text{for } x < a \\
    0, & \text{otherwise} \\
    a, & \text{for } x > b \\
\end{cases}
\end{align*}
\]

and the cumulative distribution function is

\[
\begin{align*}
F_X(x) & = \begin{cases} 
    0, & \text{for } x < a \\
    \frac{x-a}{b-a}, & \text{for } a \leq x < b \\
    1, & \text{for } x \geq b \\
\end{cases}
\end{align*}
\]

\[
E[X] = \frac{(a+b)}{2}, \text{ and } Var(X) = \frac{(b-a)^2}{12}
\]

http://en.wikipedia.org/wiki/
Some Important Continuous Random Distributions

Exponential

\[ f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & \text{for } 0 < x \end{cases} \]

and the cumulative distribution function is

\[ F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & \text{for } 0 \leq x \end{cases} \]

\[ E[X] = 1/\lambda, \text{ and } Var(X) = 1/\lambda^2 \]

http://en.wikipedia.org/wiki/
Multiple Random Variables

There are cases where the result of one experiment determines the values of several random variables.

The joint probabilities of these variables are:

Discrete variables:
\[ p(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n) \]

Continuous variables:
CDF: \[ F_{x_1x_2\ldots xn}(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n) \]

pdf: \[ f_{X_1, X_2,\ldots X_n}(x_1, x_2, \ldots x_n) = \frac{n F_{X_1, X_2,\ldots X_n}(x_1, x_2, \ldots x_n)}{x_1 x_2 \ldots x_n} \]
Independence and Conditional Probability

Independence: The random variables are said to be independent of each other when the occurrence of one does not affect the other. The pmf for discrete random variables in such a case is given by:
\[ p(x_1, x_2, \ldots, x_n) = P(X_1 = x_1)P(X_2 = x_2)\ldots P(X_n = x_n) \]
and for continuous random variables as:
\[ F_{X_1, X_2, \ldots, X_n} = F_{X_1}(x_1)F_{X_2}(x_2)\ldots F_{X_n}(x_n) \]

Conditional probability: is the probability that \( X_j = x_j \) given that \( X_i = x_i \). Then for discrete random variables the probability becomes:
\[
P(X_1 = x_1 | X_2 = x_2, \ldots, X_n = x_n) = \frac{P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}{P(X_2 = x_2, \ldots, X_n = x_n)}
\]
and for continuous random variables it is:
\[
P(X_1 = x_1 | X_2 = x_2, \ldots, X_n = x_n) = \frac{P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}{P(X_2 = x_2, \ldots, X_n = x_n)}
\]
Bayes Theorem

A theorem concerning conditional probabilities of the form $P(X|Y)$ (read: the probability of $X$, given $Y$) is

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

where $P(X)$ and $P(Y)$ are the unconditional probabilities of $X$ and $Y$, respectively.
Important Properties of Random Variables

Sum property of the expected value

Expected value of the sum of random variables:

\[ E \sum_{i=1}^{n} a_i X_i = a_i E[X_i] \]

Product property of the expected value

Expected value of product of stochastically independent random variables

\[ E \prod_{i=1}^{n} X_i = E[X_i] \]
**Important Properties of Random Variables**

Sum property of the variance

Variance of the sum of random variables is

\[
Var \sum_{i=1}^{n} a_i X_i = a_i^2 Var(X_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_i a_j \text{cov}(X_i, X_j)
\]

where \( \text{cov}(X_i, X_j) \) is the covariance of random variables \( X_i \) and \( X_j \) and

\[
\text{cov}(X_i, X_j) = E[(X_i - E[X_i])(X_j - E[X_j])]
\]

\[
E[X_i X_j] = E[X_i]E[X_j]
\]

If random variables are independent of each other, i.e., \( \text{cov}(X_i, X_j) = 0 \), then

\[
Var \sum_{i=1}^{n} a_i X_i = a_i^2 Var(X_i)
\]
Important Properties of Random Variables

**Distribution of sum** - For continuous random variables with joint pdf $f_{XY}(x, y)$ and if $Z = \Phi(X, Y)$, the distribution of $Z$ may be written as

$$F_Z(z) \ P(Z \leq z) = \int_{xy} f_{XY}(x, y) \ dx \ dy$$

where $\Phi_Z$ is a subset of $Z$.

For a special case $Z = X + Y$

$$F_Z(z) \ f_{XY}(x, y) \ dx \ dy = \int_{xy} f_{XY}(x, y) \ dx \ dy$$

If $X$ and $Y$ are independent variables, the $f_{XY}(x, y) = f_X(x)f_Y(y)$

$$f_Z(z) = f_X(x)f_Y(z-x) \ dx, \ \text{for} \ -z \leq z$$

If both $X$ and $Y$ are non-negative random variables, then pdf is the convolution of the individual pdfs, $f_X(x)$ and $f_Y(y)$

$$f_Z(z) = f_X(x)f_Y(z-x) \ dx, \ \text{for} \ 0 \leq z$$
Central Limit Theorem

The Central Limit Theorem states that whenever a random sample \((X_1, X_2, \ldots, X_n)\) of size \(n\) is taken from any distribution with expected value \(E[X_i] = \mu\) and variance \(Var(X_i) = \sigma^2\), where \(i = 1, 2, \ldots, n\), then their arithmetic mean is defined by

\[
S_n = \frac{1}{n} \sum_{i=1}^{n} X_i
\]
Central Limit Theorem

Mean of a sufficiently large number of independent random variables, each with finite mean and variance, will be approximately normally distributed.

The sample mean is approximated to a normal distribution with

\[ E[S_n] = \mu \quad \text{and} \quad Var(S_n) = \frac{\sigma^2}{n} \]

The larger the value of the sample size \( n \), the better the approximation to the normal.

This is very useful when inference between signals needs to be considered.
Poisson Arrival Model

Events occur continuously and **independently of one another**
A Poisson process is a sequence of events “randomly spaced in time”
For example, customers arriving at a bank and Geiger counter clicks are similar to packets arriving at a buffer
The rate of a Poisson process is the average number of events per unit time (over a long time)
Properties of a Poisson Process

Properties of a Poisson process

For a time interval $[0, t]$, the probability of $n$ arrivals in $t$ units of time is

$$P_n(t) = \frac{(t)^n}{n!} e^{-t}$$

For two disjoint (non overlapping) intervals $(t_1, t_2)$ and $(t_3, t_4)$, (i.e. $t_1 < t_2 < t_3 < t_4$), the number of arrivals in $(t_1, t_2)$ is independent of arrivals in $(t_3, t_4)$
Interarrival Times of Poisson Process

Interarrival times of a Poisson process

We pick an arbitrary starting point $t_0$ in time. Let $T_I$ be the time until the next arrival. We have

$$P(T_I > t) = P_0(t) = e^{-t}$$

Thus the cumulative distribution function of $T_I$ is given by

$$F_{T_I}(t) = P(T_I \leq t) = 1 - e^{-t}$$

The pdf of $T_I$ is given by

$$f_{T_I}(t) = e^{-t}$$

Therefore, $T_I$ has an exponential distribution with mean rate
Exponential Distribution

Similarly $T_2$ is the time between first and second arrivals, we define $T_3$ as the time between the second and third arrivals, $T_4$ as the time between the third and fourth arrivals and so on.

The random variables $T_1, T_2, T_3, \ldots$ are called the interarrival times of the Poisson process.

$T_1, T_2, T_3, \ldots$ are independent of each other and each has the same exponential distribution with mean arrival rate.
Memoryless and Merging Properties

Memoryless property

A random variable \( X \) has the property that “the future is independent of the past” i.e., the fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen.

Merging property

If we merge \( n \) Poisson processes with distributions for the inter arrival times

\[
1 - e^{-\lambda t} \quad \text{for } i = 1, 2, \ldots, n
\]

into one single process, then the result is a Poisson process for which the inter arrival times have the distribution \( 1 - e^{-t} \) with mean

\[
= 1 + 2 + \ldots + n
\]
Basic Queuing Systems

What is queuing theory?

Queuing theory is the study of queues (sometimes called waiting lines)

Can be used to describe real world queues, or more abstract queues, found in many branches of computer science, such as operating systems

Basic queuing theory

Queuing theory is divided into 3 main sections:

Traffic flow
Scheduling
Facility design and employee allocation
Kendall’s Notation

D.G. Kendall in 1951 proposed a standard notation for classifying queuing systems into different types. Accordingly the systems were described by the notation A/B/C/D/E where:

<table>
<thead>
<tr>
<th>A</th>
<th>Distribution of inter arrival times of customers</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>Distribution of service times</td>
</tr>
<tr>
<td>C</td>
<td>Number of servers</td>
</tr>
<tr>
<td>D</td>
<td>Maximum number of customers in the system</td>
</tr>
<tr>
<td>E</td>
<td>Calling population size</td>
</tr>
</tbody>
</table>
Kendall’s notation

A and B can take any of the following distributions types:

<table>
<thead>
<tr>
<th></th>
<th>Exponential distribution (Markovian)</th>
<th>Degenerate (or deterministic) distribution</th>
<th>Erlang distribution ((k = \text{shape parameter}))</th>
<th>Hyper exponential with parameter (k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(E_k)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(H_k)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Little’s Law

Assuming a queuing environment to be operating in a stable steady state where all initial transients have vanished, the key parameters characterizing the system are:

- the mean steady state consumer arrival

\( N \), the average no. of customers in the system

\( T \), the mean time spent by each customer in the system

which gives

\[ N = T \]
Markov Process

A Markov process is one in which the next state of the process depends only on the present state, irrespective of any previous states taken by the process.

The knowledge of the current state and the transition probabilities from this state allows us to predict the next state.
Birth-Death Process

Special type of Markov process

Often used to model a population (or, number of jobs in a queue)

If, at some time, the population has $n$ entities ($n$ jobs in a queue), then birth of another entity (arrival of another job) causes the state to change to $n+1$

On the other hand, a death (a job removed from the queue for service) would cause the state to change to $n-1$

Any state transitions can be made only to one of the two neighboring states

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State Transition Diagram

The state transition diagram of the continuous birth-death process.

P(0)  P(1)  P(2)  Probability in a state  P(n-1)  P(n)  P(n+1)
**M/M/1/ or M/M/1 Queuing System**

<table>
<thead>
<tr>
<th>Feature</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution of inter arrival times of customers</td>
<td>M</td>
</tr>
<tr>
<td>Distribution of service times</td>
<td>M</td>
</tr>
<tr>
<td>Number of servers</td>
<td>1</td>
</tr>
<tr>
<td>Maximum number of customers in the system</td>
<td></td>
</tr>
<tr>
<td>When a customer arrives in this system</td>
<td>it will be served if the server is free, otherwise the customer is queued.</td>
</tr>
<tr>
<td>In this system, customers arrive</td>
<td>according to a Poisson distribution and compete for the service in a FIFO (first in first out) manner.</td>
</tr>
<tr>
<td>Service times</td>
<td>are independent identically distributed (IID) random variables, the common distribution being exponential.</td>
</tr>
</tbody>
</table>
**Queuing Model and State Transition Diagram**

The state transition diagram of the **M/M/1/ queuing system**

Probability in a state
Equilibrium State Equations

If mean arrival rate is $\lambda$ and mean service rate is $\mu$, $i = 0, 1, 2$ be the number of customers in the system and $P(i)$ be the state probability of the system having $i$ customers.

From the state transition diagram, the equilibrium state equations are given by

$$P(0) \quad \begin{pmatrix} & & \end{pmatrix} \quad P(1), \quad i = 0,$$

$$\begin{pmatrix} & & \end{pmatrix} \quad P(i) \quad P(i + 1) \quad P(i + 1), \quad i = 1,$$

$$P(i) \quad \begin{pmatrix} & \end{pmatrix} \quad \hat{\lambda} P(0), \quad i = 1.$$
Traffic Intensity

We know that the $P(0)$ is the probability of server being free. Since $P(0) > 0$, the necessary condition for a system being in steady state is,

$$1$$

This means that the arrival rate cannot be more than the service rate, otherwise an infinite queue will form and jobs will experience infinite service time.
Queuing System Metrics

\[ 1 - P(0), \] is the probability of the server being busy. Therefore, we have

\[ P(i) = i(1-\ ) \]

The average number of customers in the system is

\[ L_s = \]

The average dwell time of customers is

\[ W_s = \frac{1}{\quad} \]
Queuing System Metrics

The average queuing length is

\[ L_q \]

\[ (i-1)P(i) \]

\[ i \quad 1 \]

\[ 1 \quad ( ) \]

The average waiting time of customers is

\[ W_q \]

\[ \frac{L_q}{2} \]

\[ (1) \quad ( ) \]
M/M/S/ Queuing Model
State Transition Diagram

0 1 2 …… S-1 S S+1 …

2 3 (S-1) S S S
# Queuing System Metrics

The average number of customers in the system is

\[ L_s = \sum_{i=0}^{s} i P(i) \]

\[ = \frac{s P(0)}{S!(1 + \frac{1}{S})^2} \]

The average dwell time of a customer in the system is given by

\[ W_s = \frac{L_s}{s} \]

\[ = \frac{1}{S S!(1 + \frac{1}{S})^2} \]
Queuing System Metrics

The average queue length is

\[
L_q (i, S)P(i) \\
\text{is} \\
\frac{S^1P(0)}{(S-1)!(S)^2}
\]

The average waiting time of customers is

\[
W_q \\
\frac{L_q}{S^P(0)} \\
S.S!(1)^2
\]
M/G/1/ Queuing Model

We consider a single server queuing system whose arrival process is Poisson with mean arrival rate. Service times are independent and identically distributed with distribution function $F_B$ and pdf $f_b$. Jobs are scheduled for service as FIFO.
Basic Queuing Model

Let $N(t)$ denote the number of jobs in the system (those in queue plus in service) at time $t$.

Let $t_n \ (n= 1, 2,..)$ be the time of departure of the $n^{th}$ job and $X_n$ be the number of jobs in the system at time $t_n$, so that

$$X_n = N(t_n), \text{ for } n =1, 2,..$$

The stochastic process can be modeled as a discrete Markov chain known as imbedded Markov chain, which helps convert a non-Markovian problem into a Markovian one.
Queueing System Metrics

The average number of jobs in the system, in the steady state is

\[ E[N] = \frac{E[\hat{B}]}{2} \]

The average dwell time of customers in the system is

\[ W_s = E[N] \left(1 + \frac{E[B]}{2} \right) \]

The average waiting time of customers in the queue is

\[ E[N] \cdot W_q \]

Average waiting time of customers in the queue is

\[ W_q = \frac{E[B]}{2(1 + \frac{E[B]}{2})} \]

The average queue length is

\[ L_q = \frac{2E[B]^2}{2} \]
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